Journal of Engineering Physics and Thermophysics, Vol. 83, No. 4, 2010

NEW INTEGRAL RELATIONS FOR ANALYTICAL SOLUTIONS OF PARABOLIC-TYPE EQUATIONS IN NONCYLINDRICAL DOMAINS

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UDC 536.2.001.24

The theory of the method of Green's functions in solving boundary-value problems of nonstationary heat conduction in domains with moving boundaries has been developed. A modification of the thermal-potential method for a uniform law of motion of the boundary has been proposed, which leads to integral relations of a new (simplest) form compared to the existing results; this makes it possible to consider numerous particular cases that are of practical interest for many applications. A number of special features of model representations of nonstationary heat transfer in domains with moving boundaries have been revealed.

Keywords: *nonstationary heat conduction, thermal potentials, Green's functions, domains with moving boundaries.*

Introduction. The scientific community of many countries is celebrating the birth centenary of the outstanding thermal physicist Academician Aleksei Vasilievich Luikov in 2010. The scientist's research work strikes one with its versatility and depth. To mark A. V. Luikov's 90th birth anniversary in 2000, a detailed paper [1] was published, devoted to his life and work, and also to his enormous scientific heritage. The 4th International Forum on Heat and Mass Transfer was devoted to this date, too.

When he was director of the Institute of Power Engineering of the BSSR Academy of Sciences since 1956 (today A. V. Luikov Heat and Mass Transfer Institute of the National Academy of Sciences of Belarus), Aleksei Vasilievich founded "Inzhenerno-Fizicheskii Zhurnal" ("Journal of Engineering Physics and Thermophysics") in 1958, whose editor-in-chief he remained till the last day of his life. It is hard to overestimate the journal's influence on the unification of thermal physicists of all levels and its role in the development of thermal physics through publications by A. V. Luikov himself and by his numerous disciples and followers. Thousands of young scientists managed to achieve their personal fulfillment in it and get a start in science. Indeed, until now the A. V. Luikov Heat and Mass Transfer Institute and "Inzhenerno-Fizicheskii Zhurnal" are perceived as one entity and are playing a leading role in sustaining and developing the world science of thermal physics.

While working on the general problems of heat and mass transfer, Aleksei Vasilievich was, in particular, involved in the study of the analytical theory of heat conduction of solids. Having an excellent command of the methods of mathematical physics, he developed efficient techniques to solve boundary-value problems of nonstationary heat conduction and related processes. A. V. Luikov is the author of the well-known monograph "Theory of Heat Conduction" (in 1969, the Book was given the USSR's highest award in the field of thermal physics — I. I. Polzunov Prize), which has remained to date a source of learning for whole generations of heat engineers. A. V. Luikov's fundamental reviews (pioneering for the domestic press) on analytical methods of solution of linear boundary-value problems [2] and nonlinear equations [3] of nonstationary heat conduction were published in 1969–1970. Thus, in [2], Aleksei Vasilievich dealt with the method of Green's functions and the thermal-potential method in solving boundary-value problems of nonstationary heat conduction in canonical-type cylindrical domains. Noting the advantages and drawbacks of the indicated approaches, A. V. Luikov pointed to the problem of extending these methods to noncylindrical domains, i.e., to domains with boundaries moving with time. It is precisely this review that has stimulated our investigations on development of the theory of the method of Green's functions for noncylindrical domains and on modification of the thermal-potential method for domains with moving boundaries. As was pointed in [2], the latter "is distinguished by a certain complexity and cumbersomeness, although it is indispensable for domains with moving boundaries." The ther-

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mal-potential theory presented below does not involve the necessity of solving integral equations within the framework of the universally adopted classical approach to application of potentials. When one uses this theory for finding analytical solutions of thermal problems and the corresponding Green functions, quite simple functions convenient for numerical calculations are obtained.

The range of problems in considering which one has to solve boundary-value problems of nonstationary heat conduction in the domains $[y_1(t), y_2(t)]$, $[0, y(t)]$, and $[y(t), \infty)]$, $t \geq 0$ ($y(t)$ are the continuous functions) is very wide. Such problems arise in studying theoretically energy-transfer processes associated with the change in the state of aggregation of a substance, in the theory of strength, dam theory, soil mechanics, electrostatic problems, filtration problems, vibration theory, the theory of zone refining of materials, the kinetic theory of crystal growth, thermomechanics in studying thermal shock, etc [4]. From the mathematical viewpoint, boundary-value transfer problems in a domain with moving boundaries are fundamentally different from classical ones. Classical methods of mathematical-physics equations are inapplicable to this class of problems because of the time dependence of the domain's boundary, since one cannot coordinate the solutions of the heat-conduction equation with the motion of the domain's boundary, remaining within the framework of these methods. Development of new approaches or modification of the existing ones for domains with moving boundaries is a natural way out.

1. Method of Green's Functions for Parabolic-Type Equations in Noncylindrical Domainss. One of the most efficient approaches for domainss with moving boundaries (noncylindrical domains) is the method of Green's functions. The method assumes preconsideration of a simple model in finding the corresponding influence function (Green's function) and makes it possible to obtain the integral representation of analytical solutions of a wide class of nonstationary-transfer problems depending on inhomogeneities in the initial formulation of the problem. However, for noncylindrical domains, singularities characteristic of the presence of moving boundaries appear. We first dwell briefly on the indicated methods for cylindrical (classical) domains.

Let D be a finite or partially bounded convex domain of variability $M(x, y, z)$, S be a piecewise continuous surface bounding the domain D, $\mathbf{n}$ be the external normal to S, $\Omega = (M \in D,\ t > 0)$ be a cylindrical domain in the phase space (x, y, z, t) with base D at $t = 0$, and $T(M, t)$ be a temperature function satisfying the conditions of the problem

$$\frac{\partial T}{\partial t} = a \Delta T(M, t) + f(M, t), \quad M \in D, \quad t > 0; \tag{1}$$

$$T(M, t)\big|_{t=0} = \Phi_0(M), \quad M \in \overline{D}; \tag{2}$$

$$\beta_1 \frac{\partial T(M, t)}{\partial n} + \beta_2 T(M, t) = \varphi(M, t), \quad M \in S, \quad t \geq 0. \tag{3}$$

Here we have

$$f(M, t) \in C^0(\Omega); \quad \Phi_0(M) \in C^1(\overline{\Omega}); \quad \varphi(M, t) \in C^0(St \geq 0); \quad \overline{D} = D + S, \quad \overline{\Omega} = (M \in \overline{D}, t \geq 0).$$

The sought solution is

$$T(M, t) \in C^2(\Omega) \cap C^0(\Omega); \quad \text{grad}_M T(M, t) \in C^0(\overline{\Omega}); \quad \beta_1^2 + \beta_2^2 > 0.$$

By virtue of the principle of superposition, which holds for linear transfer problems, we can write the integral representation for $T(M, t)$ in the form [5]

$$T(M, t) = \iiint_D \Phi_0(P)\, G(M, P, t, \tau)\big|_{\tau=0}\, dV_p + a \int_0^t \iint_S \left(G \frac{\partial T}{\partial n_p} - T \frac{\partial G}{\partial n_p} \right)\Bigg|_{P \in S} d\tau d\sigma_p$$

$$+\int_0^t \iiint_D f(P,\tau)\, G(M,t,P,\tau)\, d\tau dV_p\,, \tag{4}$$

if we know the corresponding Green function $G(M, t, P, \tau)$ for this domain as the solution of a simpler problem for the homogeneous equation (1) with homogeneous boundary conditions of the same kind as (3):

$$\frac{\partial G}{\partial t} = a\Delta_M G(M,t,P,\tau)\,, \quad M \in D\,, \quad t > \tau\,; \tag{5}$$

$$G(M,t,P,\tau)\big|_{t=\tau} = \delta(M,P)\,, \quad M \in D\,, \quad P \in D\,; \tag{6}$$

$$\beta_1 \frac{\partial G(M,t,P,\tau)}{\partial n_M} + \beta_2 G(M,t,P,\tau) = 0\,, \quad M \in S\,, \quad t > \tau\,. \tag{7}$$

Here $\delta(M, P)$ is the Dirac delta function. If $G(M, t, P, \tau)$ is considered as a function of the point P and the time τ, for G to be found we must solve the equivalent problem for the adjoint of (5)

$$\frac{\partial G}{\partial \tau} = -a\Delta_p G(M,t,P,\tau)\,, \quad P \in D\,, \quad \tau < t\,; \tag{8}$$

$$G(M,t,P,\tau)\big|_{t=\tau} = \delta(M,P)\,, \quad M \in D\,, \quad P \in D\,; \tag{9}$$

$$\beta_1 \frac{\partial G(M,t,P,\tau)}{\partial n_p} + \beta_2 G(M,t,P,\tau) = 0\,, \quad P \in S\,, \quad \tau < t\,. \tag{10}$$

If the domain D is bounded, Green's function G has the form [5]

$$G(M,t,P,\tau) = G(M,t-\tau,P) = \sum_{n=1}^{\infty} \frac{\Psi_n(M)\,\Psi_n(P)}{\|\Psi_n\|^2} \exp\left[-(\sqrt{a}\,\gamma_n)^2 (t-\tau)\right],$$

where $\Psi_n(M)$ and γ_n^2 are the eigenfunctions and eigenvalues of the corresponding (1)–(3) homogeneous problem

$$\Delta\Psi(M) + \gamma^2 \Psi(M) = 0\,, \quad M \in D\,; \quad \beta_1 \frac{\partial \Psi(M)}{\partial n} + \beta_2 \Psi(M) = 0\,, \quad M \in S\,.$$

Here $\|\Psi_n\|^2$ is the norm of eigenfunctions squared

$$\|\Psi_n\|^2 = \iiint_D \Psi_n^2(M)\, dV_M\,.$$

Let now Ω_t be a noncylindrical domain, i.e., the section of Ω_t by the characteristic plane t = const $\geq t_0 \geq 0$ be the domain D_t with boundary S_t independent of the time t. We elucidate what changes in the formulations of boundary-value problems (5)–(7) and (8)–(10) for Green's function $G(M, t, P, \tau)$ in variables (M, t) and (P, τ) (for cylindrical domains, formulation of the boundary conditions remains constant, and Eq. (5) is replaced by its ajoint (8)).

We consider the region $\Omega_t = \{y_1(t) < x < y_2(t),\ t > 0\}$ where $y_i(t)$ are continuously differentiable functions in which $T(x, t)$ satisfies the conditions

$$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2} + f(x, t), \quad y_1(t) < x < y_2(t), \quad t > 0; \tag{11}$$

$$T(x, 0) = \Phi_0(x), \quad y_1(0) \le x \le y_2(0), \quad y_i(0) \ge 0, \quad i = 1, 2; \tag{12}$$

$$[\beta_{i1}\partial T(x, t) + \beta_{i2}T(x, t)]_{x=y_i(t)} = \beta_{i3}\varphi_i(t), \quad t \ge 0, \quad i = 1, 2. \tag{13}$$

Characteristic of this problem is the presence of boundaries moving with time and consequently the circumstance that Green's function $G(x, t, x', \tau)$, by virtue of its physical meaning (heat pulse of power $Q = c\rho$ [4]), is dependent on t and τ rather than on the difference $(t - \tau)$, since it is not only the operating time of the pulse $(t - \tau)$ that will be determining but the instant τ of its generation will be determining as well. We represent $G(x, t, x', \tau)$ in the form [5]

$$G(x, t, x', \tau) = \frac{1}{2\sqrt{\pi a(t-\tau)}}\exp\left[-\frac{(x-x')^2}{4a(t-\tau)}\right] + q(x, t, x', \tau) = G_0 + q, \quad \tau < t, \tag{14}$$

where G_0 is the fundamental solution of Eq. (11) (for $f = 0$). The function G_0 satisfies the homogeneous equation (11) in variables (x, t) and the adjoint of (11) (for $f = 0$) in variables (x', τ). The function q, i.e., the regular component of Green's function (14), will be selected so as to satisfy the equation $\frac{\partial q}{\partial \tau} = -a\frac{\partial^2 q}{\partial x'^2}$ and the initial condition $q(x, t, x', \tau = t) = 0$. Hence, in accordance with (8) and (9), we have for the function $G(x, t, x', \tau)$

$$\frac{\partial G}{\partial \tau} = -a\frac{\partial^2 G}{\partial x'^2}, \quad y_1(\tau) < x' < y_2(\tau), \quad \tau < t; \tag{15}$$

$$G(x, t, x', \tau)\big|_{\tau=t} = \delta(x' - x), \quad y_1(t) < x' < y_2(t). \tag{16}$$

We consider the equality

$$\frac{\partial}{\partial \tau}[T(x', \tau)\,G(x, t, x', \tau)] = G\frac{\partial T}{\partial \tau} + T\frac{\partial G}{\partial \tau} = a\left(G\frac{\partial^2 T}{\partial x'^2} - T\frac{\partial^2 G}{\partial x'^2}\right) + Gf(x', \tau). \tag{17}$$

We integrate (17) with respect to $x' \in [y_1(\tau), y_2(\tau)]$:

$$\int_{y_1(\tau)}^{y_2(\tau)} \frac{\partial}{\partial \tau}[TG]\,dx' = a\left(G\frac{\partial T}{\partial x'} - T\frac{\partial G}{\partial x'}\right)_{x'=y_1(\tau)}^{x'=y_2(\tau)} + \int_{y_1(\tau)}^{y_2(\tau)} G(x, t, x', \tau)\,f(x', \tau)\,dx'. \tag{18}$$

Relation (18) holds for all $\tau < t$; therefore, it can be integrated with respect to τ for $0 < \tau < t - \varepsilon$, where $\varepsilon > 0$ is a number as small as desired (at $0 < \tau < t - \varepsilon$, the integrands in (18) are quite regular, since the singularity of the function G at the point $x = x'$ for $\tau = t$ is excluded). As a result we obtain

$$\int_0^{t-\varepsilon} d\tau \int_{y_1(\tau)}^{y_2(\tau)} \frac{\partial}{\partial \tau}(TG)\,dx' = a\int_0^{t-\varepsilon}\left(G\frac{\partial T}{\partial x'} - T\frac{\partial G}{\partial x'}\right)_{x'=y_2(\tau)} d\tau$$

$$-a\int_{0}^{t-\varepsilon}\left(G\frac{\partial T}{\partial x'}-T\frac{\partial G}{\partial x'}\right)_{x'=y_1(\tau)}d\tau+\int_{0}^{t}d\tau\int_{y_1(\tau)}^{y_2(\tau)}G\,(x,t,x',\tau)\,f\,(x',\tau)\,dx' .$$

Factoring the operator $\partial/\partial\tau$ on the right-hand side outside the integral sign and passing to the limit when $\varepsilon\to 0$, with account for (16) and the equality

$$\lim_{\varepsilon\to 0}\int_{y_1(t-\varepsilon)}^{y_2(t-\varepsilon)}T\,(x',t-\varepsilon)\,G\,(x,t,x',\tau)\,|_{\tau=t-\varepsilon}\,dx'=\int_{y_1(t)}^{y_2(t)}T\,(x',t)\,\delta\,(x-x')\,dx'=T\,(x,t)\,,$$

we obtain the basic integral formula, giving an idea of arbitrary solutions of Eq. (11) in the domain with moving boundaries:

$$T\,(x,t)=\int_{y_1(0)}^{y_2(0)}T\,(x',0)\,G\,(x,t,x',0)\,dx'+a\int_{0}^{t}\left[G\frac{\partial T}{\partial x'}-T\left(\frac{\partial G}{\partial x'}-\frac{1}{a}\frac{dy_2}{d\tau}G\right)\right]_{x'=y_2(\tau)}d\tau$$

$$-a\int_{0}^{t}\left[G\frac{\partial T}{\partial x'}-T\left(\frac{\partial G}{\partial x'}-\frac{1}{a}\frac{dy_1}{d\tau}G\right)\right]_{x'=y_1(\tau)}d\tau+\int_{0}^{t}\int_{y_1(\tau)}^{y_2(\tau)}G\,(x,t,x',\tau)\,f\,(x',\tau)\,dx'd\tau\,. \qquad (19)$$

Boundary solutions of the first kind (a) ($\beta_{i1}=0$, $\beta_{i2}=\beta_{i3}=1$), or the second kind (b) ($\beta_{i2}=0$, $\beta_{i1}=\beta_{i3}=1$), or the third kind (c) ($\beta_{i1}=0$, $\beta_{i2}=\beta_{i3}=(-1)^i h_i$) are assumed to be specified in (13). If the function $G(x,t,x',\tau)$ is selected so as to satisfy boundary conditions (a) in the case of the first boundary-value problem

$$G\,(x,t,x',\tau)\,|_{x'=y_i(\tau)}=0\,,\quad \tau<t\,,\quad i=1,2\,; \qquad (20)$$

conditions (b) in the case of the second boundary-value problem

$$\left.\left(\frac{\partial G}{\partial x'}-\frac{1}{a}\frac{\partial y_i}{\partial\tau}G\right)\right|_{x'=y_i(\tau)}=0\,,\quad \tau<t\,,\quad i=1,2\,; \qquad (21)$$

and conditions (c) in the case of the third boundary-value problem

$$\left.\left(\frac{\partial G}{\partial x'}+(-1)^i\left[h_i+(-1)^{i-1}\frac{1}{a}\frac{\partial y_i}{\partial\tau}\right]G\right)\right|_{x'=y_i(\tau)}=0\,,\quad \tau<t\,,\quad i=1,2\,, \qquad (22)$$

expression (18) yields the sought integral representations of $T(x,t)$ for the first boundary-value problem with conditions (a)

$$T\,(x,t)=\int_{y_1(0)}^{y_2(0)}[T\,(x',\tau)\,G\,(x,t,x',\tau)]_{\tau=0}\,dx'+a\int_{0}^{t}\left[T\,(x',\tau)\frac{\partial G}{\partial x'}\right]_{x'=y_1(\tau)}d\tau$$

$$-a\int_{0}^{t}\left[T\,(x',\tau)\frac{\partial G}{\partial x'}\right]_{x'=y_2(\tau)}d\tau+\int_{0}^{t}\int_{y_1(\tau)}^{y_2(\tau)}f\,(x',\tau)\,G\,(x,t,x',\tau)\,d\tau dx'\,; \qquad (23)$$

for the second boundary-value problem with conditions (b)

$$T(x,t)=\int\limits_{y_1(0)}^{y_2(0)}[T(x',\tau)\,G(x,t,x',\tau)]_{\tau=0}\,dx'-a\int\limits_0^t\left[\frac{\partial T(x',\tau)}{\partial x'}\,G(x,t,x',\tau)\right]_{x'=y_1(\tau)}d\tau$$

$$+a\int\limits_0^t\left[\frac{\partial T(x',\tau)}{\partial x'}\,G(x,t,x',\tau)\right]_{x'=y_2(\tau)}d\tau+\int\limits_0^t\int\limits_{y_1(\tau)}^{y_2(\tau)}f(x',\tau)\,G(x,t,x',\tau)\,d\tau dx'\,; \qquad (24)$$

and for the third boundary-value problem with conditions (c)

$$T(x,t)=\int\limits_{y_1(0)}^{y_2(0)}[T(x',\tau)\,G(x,t,x',\tau)]_{\tau=0}\,dx'-a\int\limits_0^t\left[\left(\frac{\partial T(x',\tau)}{\partial x'}-h_1T(x',\tau)\right)G(x,t,x',\tau)\right]_{x'=y_1(\tau)}d\tau$$

$$+a\int\limits_0^t\left[\left(\frac{\partial T(x',\tau)}{\partial x'}-h_2T(x',\tau)\right)G(x,t,x',\tau)\right]_{x'=y_2(\tau)}d\tau+\int\limits_0^t\int\limits_{y_1(\tau)}^{y_2(\tau)}f(x',\tau)\,G(x,t,x',\tau)\,d\tau dx'\,. \qquad (25)$$

We dwell in greater detail on consideration of the function $G(x, t, x', \tau)$ in the form (14) which satisfies (15) and (16) and one boundary condition (20)–(22) depending on the type of boundary-value problem (11)–(13).

We consider the function $G(x, t, x', \tau)$ determined by the conditions

$$\frac{\partial\overline{G}}{\partial t}=a\frac{\partial^2\overline{G}}{\partial x^2}\,,\quad y_1(t)<x<y_2(t)\,,\quad t>\tau\,; \qquad (26)$$

$$\overline{G}(x,t,x',\tau)\,|_{t=\tau}=\delta(x-x')\,,\quad y_1(\tau)<x'<y_2(\tau)\,; \qquad (27)$$

$$\left[\beta_{i1}\frac{\partial\overline{G}}{\partial x}+\beta_{i2}\overline{G}\right]_{x=y_i(t)}=0\,,\quad t>\tau\,,\quad i=1,2\,, \qquad (28)$$

where (as in (13)) we have $\beta_{i1} = 0$ for the first, $\beta_{i2} = 0$ for the second, and $\beta_{i1} = 1$ and $\beta_{i2} = \beta_{i3} = (-1)^i h_i$ for the third boundary-value problem. We show that $G(x, t, x', \tau) = \overline{G}(x, t, x', \tau)$. The proof will be performed for the second boundary-value problem; the remaining cases can be considered analogously.

We integrate the expression

$$\frac{\partial}{\partial\theta}[G(x'',\theta,x',\tau)\,G(x,t,x'',\theta)]=a\left(G\frac{\partial^2\overline{G}}{\partial x''^2}-\overline{G}\frac{\partial^2G}{\partial x''^2}\right)$$

for x'' in the interval from $y_1(\theta)$ to $y_2(\theta)$ where $t>\theta>\tau$. Taking into account the boundary conditions, we obtain for $\overline{G}$ and G

$$\frac{\partial}{\partial\theta}\int\limits_{y_1(\theta)}^{y_2(\theta)}\overline{G}G\,dx''=0\,. \qquad (29)$$

Next we integrate (29) in the interval $[\theta, t-\varepsilon]$, repeating the previous considerations (as in deriving relation (19)):

$$\int_{y_1(t-\varepsilon)}^{y_2(t-\varepsilon)} \overline{G}(x'', t-\varepsilon, x', \tau)\, [G(x, t, x'', \theta)]_{\theta=t-\varepsilon}\, dx'' = \int_{y_1(\theta)}^{y_2(\theta)} \overline{G}(x'', \theta, x', \tau)\, [G(x, t, x'', \theta)]_{\theta=t-\varepsilon}\, dx'' .$$

Passing to the limit when $\varepsilon \to 0$, we obtain

$$\int_{y_1(\theta)}^{y_2(\theta)} \overline{G}(x'', \theta, x', \tau)\, G(x, t, x'', \theta)\, dx'' = \int_{y_1(t)}^{y_2(t)} \overline{G}(x'', t, x', \tau)\, \delta(x'' - x)\, dx'' = \overline{G}(x, t, x', \tau) . \tag{30}$$

On the other hand, integration of (29) in the interval $[t+\varepsilon, \theta]$ yields

$$\int_{y_1(t+\varepsilon)}^{y_2(t+\varepsilon)} G(x, t, x'', \tau+\varepsilon)\, [\overline{G}(x'', \theta, x', \tau)]_{\theta=t+\varepsilon}\, dx'' = \int_{y_1(\theta)}^{y_2(\theta)} G(x, t, x'', \theta)\, \overline{G}(x'', \theta, x', \tau)\, dx'' .$$

Letting $\varepsilon \to 0$, we arrive at the expression

$$\int_{y_1(\theta)}^{y_2(\theta)} G(x, t, x'', \theta)\, \overline{G}(x'', \theta, x', \tau)\, dx'' = \int_{y_1(\tau)}^{y_2(\tau)} G(x, t, x'', \tau)\, \delta(x'' - x')\, dx'' = G(x, t, x', \tau) . \tag{31}$$

A comparison of (30) and (31) shows that $G(x, t, x', \tau) \equiv \overline{G}(x, t, x', \tau)$. Using the approaches of [5, 6], we can prove the existence and uniqueness of the solution of problem (26)–(28), if each curve $y_i(t)$ (i = 1 and 2) is differentiable and the derivatives $\frac{\partial y_i}{\partial t}$ satisfy the Hölder condition of $\gamma > 1/2$ order.

Thus, the function $G(x, t, x', \tau)$ can be found as the solution of the equivalent problems (15), (16), (20)–(22), and (26)–(28); in domains with moving boundaries, unlike the cylindrical domains (see (7) and (10)), no eigenvalence is retained in representation of the boundary conditions in formulations of the problems in (x, t) and (x', τ). In these cases one should be particularly cautions in finding Green's functions for the second and third boundary-value problems and in the presence of mixed boundary conditions at moving boundaries. Every case of finding Green's function of the corresponding boundary-value problem for one domain or another is very important, since it contains an ample amount of information, making it possible to write a large number of analytical solutions depending on the inhomogeneities in (11)–(13). Here, as in the case of cylindrical domains, we can also speak of the first, second, and third boundary-value problems (for the corresponding values of the coefficients β_{ij} = const in (13)). However the above equivalence in representation of the boundary conditions is not necessarily retained. In particular, the condition of heat insulation of the moving boundary of the domain $x \in [0, y(t)]$, $t \geq 0$, where $y(t)$ at $t > 0$ is a continuously differentiable function with finite derivatives of any order, has the form [5]

$$\left[\frac{\partial T(x, t)}{\partial x} + \frac{v(t)}{a} T(x, t)\right]_{x=y(t)} = 0 , \quad t > 0 , \tag{32}$$

and for the velocity of motion $v(t) = dy(t)/dt = 0$ ($y(t)$ = const), expression (32) coincides with the classical representation of the heat insulation of a stationary boundary, which follows from the Fourier law in scalar form [5].

2. Modification of the Thermal-Potential Method for Domains with a Uniformly Moving Boundary $[0, l + vt]$ and $[l + vt, \infty)$, $t \geq 0$. In [2], A. V. Luikov described the thermal potentials of the single and double layers as one possible analytical method of solution of boundary-value problems (1)–(3). In [5], this method was modified and turned out to be particularly efficient for domains with a uniformly moving boundary having numerous practical applications, in both solving problems in the initial formulation (11)–(13) (for $f = 0$ or $f = f(t)$) and constructing the corresponding Green function. The resulting analytical solutions of the problem have a new (simplest) integral form different from those known earlier.

To reduce calculations we consider, in (11)–(13), the domain $x \in [0,\ l + vt]$, $t \ge 0$, i.e., the case $y_1(t) = 0$, $y_2(t) = l + vt$ (l and v = const); also, let $\beta_{i1} = 0$ and $\beta_{i2} = \beta_{i3} = 1$ and $f(x, t) = 0$ and $\Phi_0(x) = 0$ be given. The presence of the inhomogeneities in (11)–(12) will be considered below. Also, we note that the domain $x \in [l_1 + v_1 t,\ l_2 + v_2 t]$, $t \ge 0$, for $T(x, t)$ in (11)–(13) is easily reduced to the domain $x \in [0,\ l + vt]$, $t \ge 0$, considered in our case using the transformations

$$z = x - (l_1 + v_1 t)\ , \quad T(x, t) \equiv W(z, t)\ , \quad W(z, t) = \Theta(z, t) \exp(-v_1 z/2a - v_1^2 t/4a)\ .$$

The solution of $T(x, t)$ is sought in the form of the sum of the thermal potentials [5, 6]

$$T(x, t) = \frac{\sqrt{a}}{2\sqrt{\pi}} \int_0^t \frac{\Psi_1(\tau)}{\sqrt{t-\tau}} \exp\left[-\frac{x^2}{4a(t-\tau)}\right] d\tau + \frac{\sqrt{a}}{2\sqrt{\pi}} \int_0^t \frac{\Psi_2(\tau)}{\sqrt{t-\tau}} \exp\left[-\frac{(x-l-vt)^2}{4a(t-\tau)}\right] d\tau\ , \tag{33}$$

where $\Psi_1(t)$ and $\Psi_2(t)$ are the unknown potential densities to be found. Expression (33) will be written in the space of Laplace transforms

$$\overline{T}(x, p) = \int_0^\infty \exp(-pt)\, T(x, t)\, dt\ , \quad \mathrm{Re}\, p \ge \beta > 0\ , \quad |\arg p| < \frac{\pi}{2}\ ;$$

$$\overline{T}(x, p) = \frac{\sqrt{a}}{2\sqrt{p}} \exp\left[-\frac{x}{\sqrt{a}}\sqrt{p}\right] \overline{\Psi}_1(p) + \frac{\sqrt{a}}{2\sqrt{p}} \exp\left[-\frac{l-x}{\sqrt{a}}\sqrt{p}\right] \overline{\Psi}_3[(\sqrt{p}+\gamma)^2]\ , \tag{34}$$

where $\Psi_3(t) = \Psi_2(t)\exp(\gamma^2 t)$, $\gamma^2 = \dfrac{v^2}{4a}$.

Thus, to find the inverse transform $T(x, t)$ from (34) we must seek the transforms of the densities with respect to $\overline{\Psi}_1(p)$ and $\overline{\Psi}_3[(\sqrt{p}+\gamma)^2]$. Using boundary conditions (13) in (33) (for $\beta_{i1} = 0$ and $\beta_{i2} = \beta_{i3} = 1$), we find

$$\frac{\sqrt{a}}{2\sqrt{\pi}} \int_0^t \frac{\Psi_1(\tau)}{\sqrt{t-\tau}} d\tau + \frac{\sqrt{a}}{2\sqrt{\pi}} \int_0^t \frac{\Psi_2(\tau)}{\sqrt{t-\tau}} \exp\left[-\frac{(l+v\tau)^2}{4a(t-\tau)}\right] d\tau = \varphi_1(t)\ ,$$

$$\frac{\sqrt{a}}{2\sqrt{\pi}} \int_0^t \frac{\Psi_1(\tau)}{\sqrt{t-\tau}} \exp\left[-\frac{(l+vt)^2}{4a(t-\tau)}\right] d\tau + \frac{\sqrt{a}}{2\sqrt{\pi}} \int_0^t \frac{\Psi_2(\tau)}{\sqrt{t-\tau}} \exp\left[-\frac{v^2}{4a}(t-\tau)\right] d\tau = \varphi_2(t)\ . \tag{35}$$

In the transform space, the system of integral equations (35) takes the form

$$\frac{\sqrt{a}}{2\sqrt{p}} \overline{\Psi}_1(p) + \frac{\sqrt{a}}{2\sqrt{p}} \exp\left[-\frac{l}{\sqrt{a}}\sqrt{p}\right] \overline{\Psi}_3[(\sqrt{p}+\gamma)^2] = \overline{\varphi}_1(p)\ ,$$

$$\frac{\sqrt{a}}{2\sqrt{p}} \exp\left[-\frac{l}{\sqrt{a}}(\sqrt{p}+\gamma)\right] \overline{\Psi}_1[(\sqrt{p}+\gamma)^2] + \frac{\sqrt{a}}{2\sqrt{p}} \overline{\Psi}_3(p) = \overline{\varphi}_3(p)\ , \tag{36}$$

where $\varphi_3(t) = \varphi_2(t)\exp(\gamma^2 t)$. Eliminating first $\overline{\Psi}_3(p)$ and then $\overline{\Psi}_1(p)$ from the system of functional equations (36), we find

$$\exp\left[-\frac{l}{\sqrt{a}}(\sqrt{p}+2\gamma)\right]\overline{\Psi}_1\,[(\sqrt{p}+2\gamma)^2]-\exp\left[-\frac{l}{\sqrt{a}}\sqrt{p}\right]\overline{\Psi}_1\,(p)$$

$$=\frac{2\,(\sqrt{p}+\gamma)}{\sqrt{a}}\,\overline{\varphi}_3\,[(\sqrt{p}+\gamma)^2]-\frac{2\sqrt{p}}{\sqrt{a}}\exp\left[-\frac{l}{\sqrt{a}}\sqrt{p}\right]\overline{\varphi}_1\,(p)\,, \tag{37}$$

$$\exp\left[-\frac{l}{\sqrt{a}}(\sqrt{p}+\gamma)\right]\overline{\Psi}_3\,[(\sqrt{p}+2\gamma)^2]-\exp\left[-\frac{l}{\sqrt{a}}(\sqrt{p}+\gamma)\right]\overline{\Psi}_3\,(p)$$

$$=\frac{2\,(\sqrt{p}+\gamma)}{\sqrt{a}}\,\overline{\varphi}_1\,[(\sqrt{p}+\gamma)^2]-\frac{2\sqrt{p}}{\sqrt{a}}\exp\left[-\frac{l}{\sqrt{a}}(\sqrt{p}+\gamma)\right]\overline{\varphi}_3\,(p)\,. \tag{38}$$

We introduce the notation

$$\overline{\Psi}_1\,(p^2)=\overline{A}_1\,(p)\,,\quad \overline{\Psi}_3\,(p^2)=\overline{A}_3\,(p)\,;\quad \overline{\varphi}_1\,(p^2)=\overline{F}_1\,(p)\,,\quad \overline{\varphi}_3\,(p^2)=\overline{F}_3\,(p) \tag{39}$$

and rewrite (37) and (38) in the form

$$\exp\left(-\frac{2l\gamma}{\sqrt{a}}\right)\overline{A}_1\,(p+2\gamma)-\exp\left(\frac{2l}{\sqrt{a}}\,p\right)\overline{A}_1\,(p)$$

$$=\frac{2}{\sqrt{a}}\left[(p+\gamma)\exp\left(\frac{l}{\sqrt{a}}\,p\right)\overline{F}_3\,(p+\gamma)-p\exp\left[\frac{2l}{\sqrt{a}}\,p\right]\overline{F}_1\,(p)\right], \tag{40}$$

$$\exp\left(-\frac{2l\gamma}{\sqrt{a}}\right)\overline{A}_3\,(p+2\gamma)-\exp\left(\frac{2l}{\sqrt{a}}\,p\right)\overline{A}_3\,(p)$$

$$=\frac{2}{\sqrt{a}}\left[(p+\gamma)\exp\left(\frac{l}{\sqrt{a}}\,(p-\gamma)\right)\overline{F}_1\,(p+\gamma)-p\exp\left[\frac{2l}{\sqrt{a}}\,p\right]\overline{F}_3\,(p)\right]. \tag{41}$$

Using the substitution [7]

$$\overline{A}_i\,(p)=\exp\left(\frac{l}{2\sqrt{a}\,\gamma}\,p^2\right)\overline{B}_i\,(p)\,,\quad i=1,3\,, \tag{42}$$

we reduce Eqs. (40) and (41) to equations with constant coefficients:

$$\overline{B}_1\,(p+2\gamma)-\overline{B}_1\,(p)=\frac{2}{\sqrt{a}}\exp\left(-\frac{l}{2\sqrt{a}\,\gamma}\,p^2\right)\left[(p+\gamma)\exp\left(-\frac{l}{\sqrt{a}}\,p\right)\overline{F}_3\,(p+\gamma)-p\overline{F}_1\,(p)\right], \tag{43}$$

$$\overline{B}_3\,(p+2\gamma)-\overline{B}_3\,(p)=\frac{2}{\sqrt{a}}\exp\left(-\frac{l}{2\sqrt{a}\,\gamma}\,p^2\right)\left[(p+\gamma)\exp\left(-\frac{l}{\sqrt{a}}\,(p+\gamma)\right)\overline{F}_1\,(p+\gamma)-p\overline{F}_3\,(p)\right]. \tag{44}$$

By checking directly we can assure ourselves that the sought particular solution of the functional equation $\overline{B}(p+2\gamma)-\overline{B}(p)=\overline{\Theta}(p)$ is the function $\overline{B}(p)=-\sum_{n=0}^{\infty}\overline{\Theta}\,(p+2\gamma n)$ on condition that this series converges. But a series of the

$\sum_{n=0}^{\infty} \exp\left[-\left(\frac{l}{2\sqrt{a}\,\gamma}\right)(n+\bar{d}(p))^2\right]$ type to which the solutions of Eqs. (43) and (44) are reduced converges, since $\frac{l}{2\sqrt{a}\,\gamma} > 0$. We find $\bar{B}_i(p)$, $i = 1, 3$, and next, from (39) and (42), the sought transforms of the potential densities (33) in the form (34):

$$\bar{\Psi}_1 [(\sqrt{p}+\gamma)^2] = -\frac{2}{\sqrt{a}}\left[\sum_{n=0}^{\infty} \exp\left(-\frac{2l\gamma}{\sqrt{a}}\right) n\,(n+1) - \frac{(2n+1)\,l}{\sqrt{a}}\sqrt{p}\right]$$

$$\times (\sqrt{p} + 2\,(n+1)\,\gamma)\,\bar{\varphi}_3\,[(\sqrt{p} + 2\,(n+1)\,\gamma)^2] + \frac{2}{\sqrt{a}}\sum_{n=0}^{\infty} \exp\left[-\frac{2l\gamma}{\sqrt{a}}n^2 - \frac{2nl}{\sqrt{a}}\sqrt{p}\right]$$

$$\times (\sqrt{p} + 2n\gamma)\,\bar{\varphi}_1\,[(\sqrt{p} + 2n\gamma)^2]\,, \tag{45}$$

$$\bar{\Psi}_3 [(\sqrt{p}+\gamma)^2] = -\frac{2}{\sqrt{a}}\left[\sum_{n=0}^{\infty} \exp\left(-\frac{2l\gamma}{\sqrt{a}}\right)(n+1)^2 - \frac{(2n+1)\,l}{\sqrt{a}}\sqrt{p}\right]$$

$$\times (\sqrt{p} + 2\,(n+1)\,\gamma)\,\bar{\varphi}_1\,[(\sqrt{p} + 2\,(n+1)\,\gamma)^2] + \frac{2}{\sqrt{a}}\sum_{n=0}^{\infty} \exp\left[-\frac{2l\gamma}{\sqrt{a}}n\,(n+1) - \frac{2nl}{\sqrt{a}}\sqrt{p}\right]$$

$$\times (\sqrt{p} + (2n+1)\,\gamma)\,\bar{\varphi}_3\,[(\sqrt{p} + (2n+1)\,\gamma)^2]\,. \tag{46}$$

Expressions (34), (45), and (46) yield the operational solution of the problem

$$\bar{T}(x, p) = \bar{\varphi}_1(p)\,\frac{\sqrt{a}}{2\sqrt{p}}\exp\left[-\frac{x}{\sqrt{a}}\sqrt{p}\right] + \frac{1}{\sqrt{p}}\sum_{n=1}^{\infty} \exp\left[-\frac{2l\gamma}{\sqrt{a}}n^2\right](\sqrt{p} + 2n\gamma)$$

$$\times \left\{\exp\left[-\frac{2nl + x}{\sqrt{a}}\sqrt{p}\right] - \exp\left[-\frac{2nl - x}{\sqrt{a}}\sqrt{p}\right]\right\}\bar{\varphi}_1\,[(\sqrt{p} + 2n\gamma)^2]$$

$$+ \frac{1}{\sqrt{p}}\sum_{n=0}^{\infty} \exp\left[-\frac{2l\gamma}{\sqrt{a}}n\,(n+1)\right](\sqrt{p} + (2+1)\,\gamma)$$

$$\times \left\{\exp\left[-\frac{(2n+1)\,l - x}{\sqrt{a}}\sqrt{p}\right] - \exp\left[-\frac{(2n+1)\,l + x}{\sqrt{a}}\sqrt{p}\right]\right\}\bar{\varphi}_3\,[(\sqrt{p} + (2n+1)\,\gamma)^2]\,. \tag{47}$$

Switching to the space of inverse transforms, we obtain the sought integral representation of the analytical solution $T(x, t)$:

$$T(x, t) = \frac{1}{2\sqrt{a\pi}}\sum_{n=-\infty}^{n=+\infty}\int_0^t \frac{x + 2n\,(l + vt)}{(t-\tau)^{3/2}}\,\varphi_1(\tau)\exp\left\{-\frac{v\,(l + v\tau)}{a}n^2 - \frac{[x + 2n\,(l + v\tau)]^2}{4a\,(t-\tau)}\right\}d\tau$$

$$-\frac{1}{2\sqrt{a\pi}}\sum_{n=-\infty}^{n=+\infty}\int_0^t \frac{x+(2n+1)(l+vt)}{(t-\tau)^{3/2}}\varphi_2(\tau)\exp\left\{-\frac{v(l+v\tau)}{a}n(n+1)-\frac{[x+(2n+1)(l+v\tau)]^2}{4a(t-\tau)}\right\}d\tau . \tag{48}$$

Relation (47) makes it possible to consider a number of interesting applications. One of them is construction of Green's function $G(x, t, x', \tau)$ of the first boundary-value problem for Eq. (11) with initial condition (12) in the domain in question. The function G for this case satisfies the conditions

$$\frac{\partial G}{\partial t}=a\frac{\partial^2 G}{\partial x^2}, \quad 0<x<l+vt, \quad t>\tau ; \tag{49}$$

$$G\,\big|_{t=\tau}=\delta(x-x'), \quad 0<x<l_0, \quad l_0=l+v\tau ; \tag{50}$$

$$G\,\big|_{x=0}=G\,\big|_{x=l+vt}=0, \quad t>\tau . \tag{51}$$

We represent $G(x, t, x', \tau)$ in the form [5]

$$G(x,t,x',\tau)=\frac{1}{2\sqrt{\pi a(t-\tau)}}\exp\left[-\frac{(x-x')^2}{4a(t-\tau)}\right]+q(x,t,x',\tau)=G_0(x,t,x',\tau)+q(x,t,x',\tau) , \tag{52}$$

where G_0 is the fundamental solution of the heat-conduction equation and q is the regular component of Green's function. Passing, in (49)–(51), to the function q at $t' = t-\tau$, we find

$$\frac{\partial q}{\partial t'}=a\frac{\partial^2 q}{\partial x^2}, \quad 0<x<l+vt', \quad t'>0 ; \tag{53}$$

$$q\,\big|_{t'=0}=0, \quad 0<x<l_0 ; \tag{54}$$

$$q\,\big|_{x=0}=\frac{1}{2\sqrt{\pi a t'}}\exp\left[-\frac{x'^2}{4at'}\right]=\varphi_1(t'), \quad t'>0 ; \tag{55}$$

$$q\,\big|_{x=l_0+vt'}=\frac{1}{2\sqrt{\pi a t'}}\exp\left[-\frac{(l_0-x'+vt')^2}{4at'}\right]=\varphi_2(t'), \quad t'>0 . \tag{56}$$

The transform $\overline{q}(x, p, x', \tau)$ can be written directly using relation (47), if we take into account that in the case (55), (56) the operational representation of the functions involved in (47) has the form

$$\overline{\varphi}_1(p)=-\frac{1}{2\sqrt{ap}}\exp\left(-\frac{x'}{\sqrt{a}}\sqrt{p}\right);$$

$$\overline{\varphi}_1[(\sqrt{p}+2\gamma n)^2]=-\frac{1}{2\sqrt{a}(\sqrt{p}+2\gamma n)}\exp\left(-\frac{x'}{\sqrt{a}}(\sqrt{p}+2\gamma n)\right);$$

$$\overline{\varphi}_3\,[(\sqrt{p}+(2n+1)\,\gamma^2)] = -\frac{1}{2\sqrt{a}\,(\sqrt{p}+(2n+1)\,\gamma)}\exp\left(-\frac{l_0-x'}{\sqrt{a}}+(\sqrt{p}\,(2n+1)\,\gamma)\right).$$

Substituting these functions into (47) and passing to the inverse transforms, we find the sought expression for Green's function $G(x,\ t,\ x',\ \tau)$:

$$G\,(x,t,x',\tau) = \frac{1}{2\sqrt{a\pi\,(t-\tau)}}\sum_{n=-\infty}^{n=+\infty}\exp\left(-\frac{2l_0\gamma}{\sqrt{a}}n^2-\frac{2\gamma x'}{\sqrt{a}}n\right)$$

$$\times\left\{\exp\left[-\frac{(2nl_0+x'-x)^2}{4a\,(t-\tau)}\right]-\exp\left[-\frac{(2nl_0+x'+x)^2}{4a\,(t-\tau)}\right]\right\}, \qquad (57)$$

where $l_0 = l+v\tau$ and $\gamma = \frac{v}{2\sqrt{a}}$. Investigating the obtained expression, we combine the exponents contained in the index n^2. Then a factor $\exp\,[-l_0 n^2(l+vt)/a(t-\tau)]$ will appear under the series sign. It follows that series (57) converges for negative v values, too, if we have $\tau < t < -\frac{l}{v}$. However, at $t = -\frac{l}{v}$ $(v<0)$, the prescribed domain disappears. Consequently, expression (57) can be used for any v values.

Now we can take account of the inhomogeneities in (11) and (12) and write the corresponding integral representation of the analytical solution of the first boundary-value problem in terms of Green's function (57) [5]:

$$T\,(x,t) = \int_0^l \Phi_0\,(x')\,G\,(x,t,x',\tau)\big|_{\tau=0}\,dx' + a\int_0^t \varphi_1\,(\tau)\left.\frac{\partial G}{\partial x'}\right|_{x'=0} d\tau - a\int_0^t \varphi_2\,(\tau)\left.\frac{\partial G}{\partial x'}\right|_{x'=l+vt} d\tau$$

$$+\int_0^t\int_0^{l+v\tau} F\,(x',\tau)\,G\,(x,t,x',\tau)\,d\tau dx'\,.$$

Expression (57) enables us to write Green's function $G(r,\ t,\ r',\ \tau)$ of the first boundary-value problem for the centrally symmetric domain $r\in[0,\ R+vt]$, $t\geq 0$, if we take into account that this case is reduced, with the substitution $\Theta = rG$, to the case studied in the Cartesian coordinate system $(r,\ t)$:

$$G\,(r,t,r',\tau) = \frac{1}{8\pi r r'\sqrt{a\pi\,(t-\tau)}}\sum_{n=-\infty}^{n=+\infty}\exp\left(-\frac{R_0 v}{a}n^2-\frac{r'v}{a}n\right)$$

$$\times\sum_{k=1}^{2}(-1)^{k+1}\exp\left[-\frac{(2nR_0+r'+(-1)^k\,r)^2}{4a\,(t-\tau)}\right],$$

where $R_0 = R+v\tau$. For the remaining boundary conditions, the special features of the method are only in solving the finite-difference equation and passing to the inverse transform. Thus, for boundary conditions of the mixed type in (13) $\beta_{11} = \beta_{22} = 0$, $\beta_{12} = \beta_{13} = \beta_{23} = \beta_{21} = 1$ and $y(t) = vt$ $(v>0)$, the corresponding Green function has the form

$$G\,(x,t,x',\tau) = \frac{1}{2\sqrt{a\pi\,(t-\tau)}}\sum_{n=-\infty}^{n=+\infty}(-1)^n\exp\left(-\frac{v^2\tau}{a}n^2-\frac{vx'}{a}n\right)$$

$$\times\left\{\exp\left[-\frac{(2nv\tau - x + x')^2}{4a(t-\tau)}\right] - \exp\left[-\frac{(2nv\tau + x + x')^2}{4a(t-\tau)}\right]\right\}$$

$$+\frac{v}{2a}\sum_{n=-\infty}^{n=+\infty}(-1)^n n \exp\left(-\frac{v^2\tau}{a}n^2 - \frac{vx'}{a}n\right)\left[\Phi\left(-\frac{2nv\tau + x + x'}{2\sqrt{a(t-\tau)}}\right) - \Phi\left(\frac{2nv\tau + x' - x}{2\sqrt{a(t-\tau)}}\right)\right],$$

where $\Phi(z) = \frac{2}{\sqrt{\pi}}\int_0^z \exp(-y^2)dy$ is the Laplace function.

The method can also be extended to some other domains and laws of motion of the boundary. It becomes particularly efficient for the domain $x \geq l + vt$, $t \geq 0$. The solution of the problem

$$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2}, \quad x > l + vt, \quad t > 0; \tag{58}$$

$$T(x, 0) = 0, \quad x \geq l; \quad (\beta_1 \partial T/\partial x + \beta_2 T)_{x=l+vt} = \beta_3 \varphi(t), \quad t \geq 0; \tag{59}$$

$$|T(x, t)| < +\infty, \quad x \geq l + vt, \quad t \geq 0 \tag{60}$$

is written in the form of a generalized thermal potential of the single layer along the curve $x = l + vt$:

$$T(x, t) = \frac{\sqrt{a}}{2\sqrt{\pi}}\int_0^t \frac{\Psi(\tau)}{\sqrt{t-\tau}}\exp\left[-\frac{(x - l - v\tau)^2}{4a(t-\tau)}\right]d\tau, \tag{61}$$

where $\Psi(t)$ is the form of the unknown potential density to be found from boundary condition (59). In the space of Laplace transforms $\overline{T}(x, p) = \int_0^\infty T(x, t)\exp(-pt)dt$, Re $p \geq \beta > 0$, $|\arg p| < \frac{\pi}{2}$, expression (61) takes the form

$$\overline{T}(x, p) = \frac{\sqrt{a}}{2\sqrt{p}}\exp\left[-(x - l)\sqrt{p/a}\right]\overline{\Psi}\left(p - v\sqrt{p/a}\right), \tag{62}$$

whence it follows that the unknown density must be found in the form $\overline{\Psi}(p - v\sqrt{p/a})$. The final operational (basic) solution of problem (58)–(60) has the form

$$\overline{T}(x, p) = \overline{\Theta}(p)\left(1 - \frac{v/(2a)}{\sqrt{p/a}}\right)\exp\left[-(x - l)\sqrt{p/a}\right]\overline{\varphi}\left(p - v\sqrt{p/a}\right), \tag{63}$$

where

$$\overline{\Theta}(p) = \begin{cases} 1 & (\beta_1 = 0, \ \beta_2 = \beta_3 = 1) \quad \text{for the 1st boundary–value problem;} \\ -1/\sqrt{p/a} & (\beta_2 = 0, \ \beta_1 = \beta_3 = 1) \quad \text{for the 2nd boundary–value problem;} \\ h/(h + \sqrt{p/a}) & (\beta_1 = 0, \ \beta_2 = \beta_3 = -h) \quad \text{for the 3rd boundary–value problem.} \end{cases}$$

Expression (63) involves numerous particular cases of the boundary function $\varphi(t)$ in (59), which are of practical interest. This function can be homogeneous, impulse, pulsating, periodic, etc. Passage to inverse transforms follows the well-known rules of operational calculus and leads to analytical solutions of a very compact form. Thus, for the first boundary-value problem ($T(x, t)|_{x=l+vt} = \varphi(t)$), we have

$$T(x,t)=\frac{\sqrt{a}}{2\sqrt{\pi}}\int_0^t \frac{x-(l+vt)}{(t-\tau)^{3/2}}\,\varphi(\tau)\exp\left[-\frac{(x-l-v\tau)^2}{4a(t-\tau)}\right]d\tau\,;$$

for the second boundary-value problem $\left(\left.\frac{\partial T}{\partial x}\right|_{x=l+vt}=\varphi(t)\right)$, we obtain

$$T(x,t)=\frac{v}{a}\int_0^t \varphi(\tau)\,\Phi^*\left(-\frac{x-l-v\tau}{2\sqrt{a(t-\tau)}}\right)d\tau-\frac{\sqrt{a}}{\sqrt{\pi}}\int_0^t \frac{\varphi(\tau)}{\sqrt{t-\tau}}\exp\left[-\frac{(x-l-v\tau)^2}{4a(t-\tau)}\right]d\tau\,;$$

and for the third boundary-value problem $\left(\left.\frac{\partial T}{\partial x}\right|_{x=l+vt}=h(T|_{x=l+vt}-\varphi(t))\right)$, we have

$$T(x,t)=\frac{h\sqrt{a}}{\sqrt{\pi}}\int_0^t \frac{\varphi(\tau)}{\sqrt{t-\tau}}\exp\left[-\frac{(x-l-v\tau)^2}{4a(t-\tau)}\right]d\tau$$

$$-\,ah\left(h+\frac{v}{a}\right)\int_0^t \varphi(\tau)\,\Phi^*\left(-\frac{x-l-v\tau}{2\sqrt{a(t-\tau)}}+h\sqrt{a(t-\tau)}\right)\exp\left[-(x-l-v\tau)\,h+ah^2(t-\tau)\right]d\tau\,,$$

where $\Phi^*(z)=1-\Phi(z)$.

All considerations hold in the presence of a homogeneous nonstationary source $f(t)$ in (58), too. In this case on the right-hand side of (61), there appear a term $\int_0^1 f(\tau)d\tau$ slightly complicating the calculations.

The most difficult case is to find the analytical solution of the problem in $\overline{\Theta}_t=\{x\in[l+vt,\infty),\ t\geq 0\}$ in the presence, in the initial formulation, of nonhomogeneous and nonstationary boundary functions

$$\frac{\partial T}{\partial t}=a\frac{\partial^2 T}{\partial x^2}+f(x,t)\,,\quad x>l+vt\,,\quad t>0\,; \tag{64}$$

$$T(x,t)|_{t=0}=\Phi_0(x)\,,\quad x\geq l\,; \tag{65}$$

$$\left.\left(\beta_1\frac{\partial T}{\partial x}+\beta_2 T\right)\right|_{x=l+vt}=\beta_3\varphi(t)\,,\quad t>0\,; \tag{66}$$

$$|T(x,t)|<+\infty\,,\quad x\geq l+vt\,,\quad t\geq 0\,. \tag{67}$$

Here $f(x,t)\in C^0(\overline{\Omega}_t)$, $\Phi_0(x)\in C^1(\overline{\Omega}_t)$, and $\varphi(t)\in C^0[0;+\infty)$; the sought solution is $T(x,t)\in C^2(\Omega_t)\cap C^0(\overline{\Omega}_t)$; $\mathrm{grad}_x\,T(x,t)\in C^0(\overline{\Omega}_t)$. In this case it is expedient to write the integral representation of the solution of the problem in terms of Green's function $G(x,t,x',\tau)$. Expression (63) enables us to do this with minimum calculations. The latter will be considered in detail. According to (49)–(51), Green's function for boundary-value problem (64)–(67) is found from the conditions

$$\frac{\partial G}{\partial t}=a\frac{\partial^2 G}{\partial x^2}\,,\quad x>l+vt\,,\quad t>\tau\,; \tag{68}$$

$$\left. G(x, t, x', \tau) \right|_{t=\tau} = \delta(x - x'), \quad x > l + v\tau, \quad x' > l + v\tau; \tag{69}$$

$$\left. \left(\beta_1 \frac{\partial G}{\partial x} + \beta_2 G \right) \right|_{x=l+vt} = 0, \quad t > \tau; \tag{70}$$

$$\left| G(x, t, x', \tau) \right| < +\infty, \quad x \geq l + v\tau, \quad t \geq \tau, \tag{71}$$

where $\beta_1 = 0$ and $\beta_2 = 1$ in the case of the 1st boundary-value problem, $\beta_1 = 1$ and $\beta_2 = 1$ in the case of the 2nd boundary-value problem, and $\beta_1 = 1$ and $\beta_2 = -h$ in the case of the 3rd boundary-value problem. We find G for the 3rd boundary-value problem, since the remaining cases follow from the latter. According to (14), we seek Green's function in the form of the sum of the fundamental solution of Eq. (68) and the regular component

$$G(x, t, x', \tau) = \frac{1}{2\sqrt{\pi a (t - \tau)}} \exp\left[-\frac{(x - x')^2}{4a (t - \tau)} \right] + q(x, t, x', \tau). \tag{72}$$

In the case of the 3rd boundary-value problem for the function $q(x, t', x', \tau)$, according to (72), we have

$$\frac{\partial q}{\partial t'} = a \frac{\partial^2 q}{\partial x^2}, \quad x > l_0 + vt', \quad t' > 0; \tag{73}$$

$$\left. q(x, t', x', \tau) \right|_{t'=0} = 0, \quad x > l_0; \tag{74}$$

$$\left. \frac{\partial q}{\partial x} \right|_{x=l_0+vt'} = h \left\{ \left. q \right|_{x=l_0+vt'} - \frac{(x' - l_0) - (v + 2ah) t'}{4h\sqrt{\pi} (at')^{3/2}} \exp\left[-\frac{(l_0 + vt' - x')^2}{4at'} \right] \right\}, \quad t' > 0; \tag{75}$$

$$\left| q(x, t', x', \tau) \right| < +\infty, \quad x > l_0 + vt', \quad t' \geq 0, \tag{76}$$

where $l_0 = l + v\tau$ and $t' = t - \tau$. Now, in accordance with approach (63), we separate, in boundary condition (75), the function

$$\varphi(t') = \frac{(x' - l_0) - (v + 2ah) t'}{4h\sqrt{\pi} (at')^{3/2}} \exp\left[-\frac{(l_0 + vt' - x')^2}{4at'} \right],$$

find its transform in the form indicated in (63):

$$\bar{\varphi}\left(p - \frac{v}{\sqrt{a}} \sqrt{p} \right) = \frac{\sqrt{p} - \left(\frac{v}{\sqrt{a}} + h\sqrt{a} \right)}{2ah \left(\sqrt{p} - \frac{v}{2\sqrt{a}} \right)} \exp\left[-\frac{x' - l_0}{\sqrt{a}} \sqrt{p} + \frac{v}{a} (x' - l_0) \right]$$

and pass, using (63), to the transform for $\bar{q}(x, p, x', \tau)$

$$\bar{q} = \frac{1}{2\sqrt{a}} \frac{\sqrt{p} - \left(\frac{v}{\sqrt{a}} + h\sqrt{a}\right)}{\sqrt{p}\,(\sqrt{p} + h/\sqrt{a})} \exp\left[-\frac{x + x' - 2l_0}{\sqrt{a}} \sqrt{p} + \frac{v}{a}(x' - l_0) \right]. \quad (77)$$

Turning to the inverse transforms in (77) and taking account of (72), we find Green's function for the 3rd boundary-value problem in the domain $x > l + vt,\ t > 0$:

$$G(x, t, x', \tau) = \frac{1}{2\sqrt{\pi a (t - \tau)}} \left\{ \exp\left[-\frac{(x - x')^2}{4a(t - \tau)} \right] + \exp\left[-\frac{(x + x' - 2(l + v\tau))^2}{4a(t - \tau)} + \frac{v}{a}(x' - (l + v\tau)) \right] \right\}$$

$$- \left(h + \frac{v}{2a}\right) \exp\left\{ [x + x' - 2(l + v\tau)\, h] + ah^2 (t - \tau) + \frac{v}{a}[x' - (l + v\tau)] \right\}$$

$$\times \Phi^* \left(\frac{x + x' - 2(l + v\tau)}{2\sqrt{a(t - \tau)}} + h\sqrt{a(t - \tau)} \right), \quad (78)$$

where $\Phi^*(z) = 1 - \Phi(z)$. Setting $h = 0$ in (77), we find Green's function for the 2nd boundary-value problem

$$G(x, t, x', \tau) = \frac{1}{2\sqrt{\pi a (t - \tau)}} \left\{ \exp\left[-\frac{(x - x')^2}{4a(t - \tau)} \right] + \exp\left[-\frac{(x + x' - 2(l + v\tau))^2}{4a(t - \tau)} + \frac{v}{a}(x' - (l + v\tau)) \right] \right\}$$

$$- \frac{v}{2a} \exp\left[\frac{v}{a}(x' - (l + v\tau)) \right] \Phi^* \left(\frac{x + x' - 2(l + v\tau)}{2\sqrt{a(t - \tau)}} \right).$$

Passage to the limit at $(1/h) \to 0$ on (77) leads to Green's function for the 1st boundary-value problem

$$G(x, t, x', \tau) = \frac{1}{2\sqrt{\pi a (t - \tau)}} \left\{ \exp\left[-\frac{(x - x')^2}{4a(t - \tau)} \right] - \exp\left[-\frac{(x + x' - 2(l + v\tau))^2}{4a(t - \tau)} + \frac{v}{a}(x' - (l + v\tau)) \right] \right\}. \quad (79)$$

The integral representation of the analytical solution of problem (64)–(67) has, according to (4), the form

$$T(x, t) = \int_l^\infty \Phi_0(x')\, G(x, t, x', 0)\, dx' + a \int_0^t \left(\gamma_1 \frac{\partial G}{\partial x'} - \gamma_2 G \right)_{x' = l + v\tau} \varphi(\tau)\, d\tau$$

$$+ \int_0^l \int_{l + v\tau}^\infty f(x', \tau)\, G(x, t, x', \tau)\, d\tau dx', \quad (80)$$

where $\gamma_1 = 1$ and $\gamma_2 = 0$ in the case of the 1st boundary-value problem, $\gamma_1 = 0$ and $\gamma_2 = 1$ in the case of the 2nd boundary-value problem, and $\gamma_1 = 0$ and $\gamma_2 = -h$ in the case of the 3rd boundary-value problem. Moving boundaries produce effects that are manifested as a graphic representation of the temperature functions. For this purpose, we consider, in (64)–(67), the 1st boundary-value problem ($\beta_1 = 0$ and $\beta_2 = \beta_3 = 1$) for $f(x, t) = f(\tau)$ and $\Phi_0(x) = T_0$. In dimensionless variables

$$z = \frac{x}{l};\ \text{Fo} = \frac{at}{l^2};\ \text{Pe} = \frac{vl}{a};\ \varphi(\text{Fo}) = \frac{\varphi(t)}{T_0};\ q(\text{Fo}) = \frac{f(t)\, l^2}{ac\rho};\ W(z, \text{Fo}) = \frac{T(x, t)}{T_0}$$

the solution of such a problem for $\varphi(\text{Fo}) = \varphi_0 = \text{const}$ and $q(\text{Fo}) = q_0 = \text{const}$ on the basis of (80) has the form

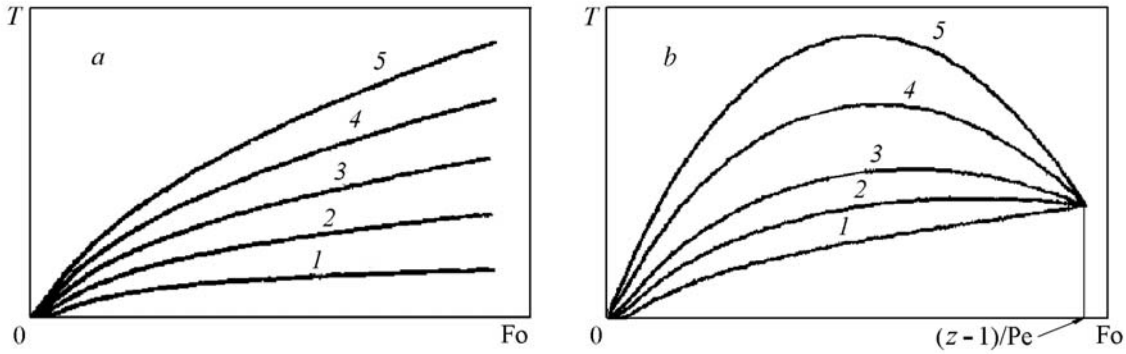

Fig. 1. Temperature function $T(z, \text{Fo})$ vs. dimensionless time Fo in the case of temperature heating with a constant temperature at the boundary and a constant-strength heat source in the cross section $z = 2$ at $\varphi_0 = 1$ and Pe = 0 (a) and Pe = 0.35 (b) for different source strengths: 1) $q = 0$, 2) 0.5, 3) 1, 4) 2, and 5) 3.

$$W(z, \text{Fo}) = \frac{\varphi_0}{2}\left\{\Phi^*\left(\frac{z-1}{2\sqrt{\text{Fo}}}\right) + \exp\left[-\text{Pe}\,(z-1-\text{Pe}\,\text{Fo})\right]\Phi^*\left(\frac{z-1}{2\sqrt{\text{Fo}}} - \text{Pe}\sqrt{\text{Fo}}\right)\right\}$$

$$-\frac{q_0\,(z-1)}{2\sqrt{\text{Pe}}}\,\Phi^*\left(\frac{z-1}{2\sqrt{\text{Fo}}}\right) + \frac{q_0\,(z-1-2\text{Pe}\,\text{Fo})}{2\text{Pe}}\exp\left[-\text{Pe}\,(z-1-2\text{Pe}\,\text{Fo})\right]\Phi^*\left(\frac{z-1}{2\sqrt{\text{Fo}}} - \text{Pe}\sqrt{\text{Fo}}\right) + q_0\text{Fo}\,. \quad (81)$$

Figure 1a plots $T(z, \text{Fo})$ versus Fo in the cross section $z = 2$ for Pe = 0, $\varphi_0 = 1$, and different q_0 values; the plots have been constructed from relation (81). The character of the curves in the figure is typical of heat exchange in the domain $z > 1$ with an internal homogeneous heat source. However the situation sharply changes, as the boundary motion appears, which is reflected in Fig. 1b constructed from relation (81), too, with the same conditions. When $z > 1 + \text{PeFo}$ we have $0 < \text{Fo} < (z-1)/\text{Pe}$ for the cross section z = const; when $z = 2$, as Fig. 1b shows, there is an instant of time in the interval $\text{Fo} \in (0, 1/\text{Pe})$ at which the temperature attains its maximum. The last circumstance is of great applied importance in thermal mechanics studying thermal shock [8], which leads to discontinuities in the internal layers of a solid.

The thermal-potential method can be efficiently used in finding the analytical solutions of comparatively new heat-conduction problems with an integral boundary condition that occur when a number of environmental, biological, plasma-physical, and thermomechanical processes are modeled [4]. Here one can obtain results of substantial interest. For example, the solution of the problem

$$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2}\,, \quad x > 2\sqrt{a}\,y(t)\,, \quad t > 0\,; \quad (82)$$

$$T(x, 0) = 0\,, \quad x \ge 0\,; \quad |T(x, t)| < +\infty\,, \quad x \ge 0\,, \quad t \ge 0\,; \quad (83)$$

$$\int_{2\sqrt{a}y(t)}^{\infty} T(x, t)\,dx = 2\sqrt{a}\,y(t)\,, \quad (84)$$

where $y(t)$ is the known time function, should be written in the form of the thermal potential

$$T(x, t) = \int_0^t \frac{\Psi(\tau)}{\sqrt{t-\tau}}\exp\left\{-\frac{[x - 2\sqrt{a}\,y(\tau)]^2}{4a\,(t-\tau)}\right\}d\tau\,,$$

whose unknown density $\Psi(\tau)$ is found from boundary condition (84), which leads to the equation

$$\int_0^1 \Psi(z\tau)\left[1-\Phi\left(\frac{y(t)-y(zt)}{\sqrt{t(1-z)}}\right)\right]dz = 2y(t)/\sqrt{\pi t}\,. \tag{85}$$

Let $y(t) = \beta t$ be given in (82). The operational solution of the integral equation (85) will have the form

$$\Psi(t) = \frac{2\beta}{\sqrt{\pi}}\left[1+\beta^2 t+\left(\beta^2 t+\frac{1}{2}\right)\Phi(\beta\sqrt{t})\,\frac{\beta\sqrt{t}}{\sqrt{\pi}}\exp(-\beta^2 t)\right].$$

When $y(t) = \beta\sqrt{t}$ Eq. (85) has the solution

$$\Psi(t) = \frac{2\beta}{\sqrt{\pi\gamma t}}\,,\quad \gamma = [1+\Phi(\beta)]\,[1-\sqrt{\pi}\,\beta\exp(\beta^2)\,\Phi^*(\beta)]\,.$$

Conclusions. Despite the apparent simplicity of mathematical models of nonstationary heat transfer in domains with boundaries moving with time, the above problems are far from trivial for obtaining their exact analytical solution. The simplest cases have practically been studied for the linear, parabolic (root), and square laws of motion of the boundary, but for these cases (in the Cartesian coordinate system), too, the analytical heat-conduction theory is still in its infancy. As far as the more complex laws of boundary motion in both the Cartesian and cylindrical (radial flow) and spherical (central symmetry) coordinate systems are concerned, we have only suggested in [4] possible ways to studying heat transfer in domains of this kind, and much work on finding analytical solutions of the corresponding heat-conduction problems, studying the properties of these solutions, and constructing temperature nomograms lies ahead. As the analysis of the literature sources in [4] shows, solution of problems of this kind brings about a wide range of problems of computational mathematics, special-function theory, and mathematical-physics methods. The situation is even more aggravated by extending boundary motion to the boundary-value problems of nonstationary heat conduction for hyperbolic-type equations on the basis of A. V. Luikov's hypothesis on the finite velocity of propagation of heat [9], to media with thermal memory [10, 11], and to deformable media with allowance for the effect of connectivity of the field of temperature and deformations in the heat-conduction equation [5]. The above cases represent a virtually undeveloped field of the analytical theory of heat conduction of solids. These investigations will be continued in the 21st century and the author wishes the readers every success in this matter.

NOTATION

a, thermal diffusivity; c, specific heat; C^0, C^1, and C^2, classes of functions, which indicate the order of a continuous derivative; Fo, Fourier number; h_i, relative coefficient of heat exchange; l, length; n, summation variable; Pe, Péclet number; p, parameter in the Laplace transformation; Q, heat-pulse power; q_0, constant heat-source strength; T, temperature; T_0, initial temperature; t, time, t_0, fixed time; x, y, z, space coordinates; $\delta(M, P)$, Dirac delta function; ε, small quantity; ρ, density; $\Phi(z)$, Laplace function.

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